Cherry Ng - Research Statement

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OVERVIEW

My research interests fall under the banner of topology, and more specifically, equivariant algebraic topology. One central question in algebraic topology involves distinguishing various spaces up to homotopy. There is a long history of computing invariants of spaces in service of this goal. For example, the Euler characteristic, computed as $\chi = (\#\,\text{of vertices}) - (\#\,\text{of edges}) + (\#\,\text{of faces})$, distinguishes between different types of polyhedra. Another popular topological invariant is the fundamental group, used to distinguish homotopy types of spaces. I am interested in two particular objects: Bredon homology, which is an equivariant topological invariant, and transfer systems, which are combinatorial objects equivalent to N_∞ operads. In this document I'll give a summary of my current research, descriptions of research with students, and plans for future work.

EQUIVARIANT HOMOLOGY

One of the most well-loved topological invariants is classical homology. Classical homology groups, denoted $H_n(X)$, are a popular topological invariant because they are good at highlighting certain geometric features. For example, if X is a closed, connected manifold of dimension n, then X is orientable precisely when the n^{th} homology group $H_n(X)$ is isomorphic to $\mathbb{Z}$.

Given the success of classical homology, it makes sense to seek an extension for spaces that also have group actions. One such equivariant homology theory that is able to mimic the nice properties of classical homology is Bredon homology. Much like its classical counterpart, it is robust and powerful. Unlike its classical counterpart though, the repertoire of completed computations is very small. This is because adding a group action to a space increases the complexity significantly. For one, an equivariant homology theory must respect the chosen group action. This means the classical homology method of decomposing a space into disks D^n is insufficient, because the disks themselves need to have a group action. So the appropriate analogue of plain disks D^n becomes whole orbits of several disks of the form $G/H \times D^n$. Additionally, the output of a homology computation is no longer a group, but a diagram of groups called a Mackey functor.

For these and other reasons, many equivariant homology results, even those for well-known spaces and groups, were only recently computed. For example, the homology of a point for dihedral groups was computed in 2020 [KL20], and the homology of a point for symmetric group Σ_4 was computed in 2018 [Zhu18]. The cohomology of surfaces with action by cyclic group C_2 [Haz23] and projective spaces [BDK23] were both computed in 2023. My thesis work in 2022 focused on the homology of a point for the nonabelian group of order 21. The core result of that work is the following theorem.

> **Theorem.** *Let G_{21} be the non-abelian group of order 21 and Λ the irreducible degree-6 representation of G_{21}. Let S^Λ be the representation sphere resulting from the one-point compactification of Λ. Then the Mackey functor-valued* [3] *integer-graded Bredon homology of S^Λ is*

$H\underline{\mathbb{Z}}_6(S^\Lambda)$
$$\begin{tikzcd}
 & \mathbb{Z} \arrow[dl, shift left] \arrow[ddr, shift left] & \\
\mathbb{Z} \arrow[ur, shift left] \arrow[ddr, shift left] & & \\
 & & \mathbb{Z} \arrow[uul, shift left] \arrow[dl, shift left] \\
 & \mathbb{Z} \arrow[uul, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

$H\underline{\mathbb{Z}}_4(S^\Lambda)$
$$\begin{tikzcd}
 & \mathbb{Z}/3 \arrow[dl, shift left] \arrow[ddr, shift left] & \\
\mathbb{Z}/7 \arrow[ur, shift left] \arrow[ddr, shift left] & & \\
 & & \mathbb{Z}/3 \arrow[uul, shift left] \arrow[dl, shift left] \\
 & 0 \arrow[uul, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

$H\underline{\mathbb{Z}}_2(S^\Lambda)$
$$\begin{tikzcd}
 & \mathbb{Z}/3 \arrow[dl, shift left] \arrow[ddr, shift left] & \\
\mathbb{Z}/7 \arrow[ur, shift left] \arrow[ddr, shift left] & & \\
 & & \mathbb{Z}/3 \arrow[uul, shift left] \arrow[dl, shift left] \\
 & 0 \arrow[uul, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

$H\underline{\mathbb{Z}}_0(S^\Lambda)$
$$\begin{tikzcd}
 & \mathbb{Z}/7 \arrow[dl, shift left] \arrow[ddr, shift left] & \\
\mathbb{Z}/7 \arrow[ur, shift left] \arrow[ddr, shift left] & & \\
 & & 0 \arrow[uul, shift left] \arrow[dl, shift left] \\
 & 0 \arrow[uul, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

> *where $H\underline{\mathbb{Z}}_n(S^\Lambda) = 0$ for any $n \neq 0, 2, 4, 6$.*

[c] Parts of the orbit category corresponding to the subgroups conjugate to C_3 are suppressed for succinctness

HOMOTOPICAL COMBINATORICS

Let X be a space with multiplication $m : X \times X \to X$. If this multiplication isn't strictly commutative, we might hope that it commutes up to homotopy. More precisely, we can swap the factors of $X \times X$ and compose with m to produce a new map n. If n and m have a homotopy between them (and all higher coherences) then m is ***homotopy commutative***. The structure of homotopy commutative operations is organized by E_∞-operads. It's known that all E_∞-operads are homotopy equivalent, which is to say that there is only one way to be homotopy commutative.

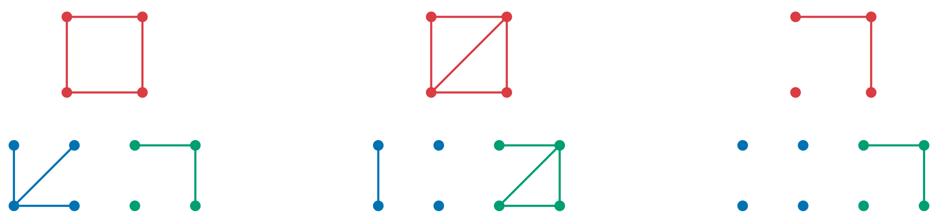
If we fix a finite group G and consider G-spaces, then the equivariant analog of an E_∞-operad is an N_∞-operad. In contrast to the non-equivariant case, N_∞-operads vary based on G, and there is ***not*** only one N_∞-operad.

Rubin [Rub19] [Rub21] showed that the homotopy theory of N_∞-operads is equivalent to the homotopy theory of a highly combinatorial object called a ***transfer system***. A transfer system in a category C is a wide subcategory of C which is closed under pullbacks by arbitrary morphisms in C. My interest in transfer systems is to study model structures on posets. The link between model structures and transfer systems is that the acyclic fibrations of a given model structure always form a transfer system.

My current project builds off the work in [BOOR23], where the authors establish several results about finite total order posets $[n] = \{0, 1, 2, \ldots, n\}$. For one, the number of possible Quillen model structures on $[n]$ whose homotopy category is isomorphic to $[k]$ is

$$\frac{2(k+1)}{n+k+2}\binom{2n+1}{n-k}$$

Intriguingly, this enumeration is accomplished by linking the fact that the number of N_∞-operads on cyclic group C_{p^n} is exactly the $(n+1)$−th Catalan number [BBR21], and that there is a bijection between the set of weak equivalence classes of N_∞-operads on C_{p^n} and the set of model structures on $[n]$ where all morphisms are weak equivalences [FOO+21]. One natural extension would be to enumerate all model structures on an $[n] \times [m]$ lattice.



Three possible model structures on $[1] \times [1]$. The red, blue, and green lines represent weak equivalences, acyclic fibrations, and acyclic cofibrations, respectively, where each map goes from a source on the lower / left side toward a target on the upper / right side.

One can modify a model structure by adding weak equivalences. If in this process we also preserve the set of acyclic fibrations (resp. acyclic cofibration), then is called ***left*** (resp. ***right***) ***Bousfield localization***. A second fact established in [BOOR23] is that every model structure on $[n]$ can be achieved via a sequence of left and right Bousfield localizations starting from the trivial model structure. My current work has uncovered that this claim does not hold for larger lattices. In fact, we can easily produce model structures on $[1] \times [2]$ that are not left or right localizations of another model structure. The work of characterizing this failure is ongoing.

STUDENT RESEARCH

Homology of projective spaces

Inspired by the recent work on projective spaces [BDK23] and the equivariant cellular decomposition discussed in my thesis [Ng22], I began a project with Eli Carroll, an undergraduate at Northwestern, to produce further equivariant homology computations for projective spaces. The methodology in [BDK23] uses a cellular filtration to decompose projective spaces and ultimately computes Bredon homology for groups G with prime power order using $\mathbb{Z}/p$-coefficients. Eli and I explored decompositions of projective spaces to compute Bredon homology for cyclic groups using constant $\mathbb{Z}$ coefficients. This ongoing work is the subject of Eli's upcoming 2025 senior thesis.

Topological data analysis

Bohai Lu, an undergraduate at Northwestern, was inspired by work he had seen about applying topological data analysis to voting patterns [Sin22] and wanted to extend this work to analyze the three-party House of Commons in Canada. In this project, Bohai learned to code and use data analysis packages in Python. He also learned the basics of the MAPPER algorithm, a process used to extract topological information from high-dimensional data. Viewing the Yeah/Nay voting record of each member of Parliament (MP) as a data point in $\mathbb{R}^n$, where n is the number of votes recorded in a year, Bohai was able to apply TDA methods for dimensionality reduction, data clustering, and data visualization to produce graphs that highlight the fractured nature of party-line voting. This work is ongoing and will conclude in 2025.

FUTURE WORK WITH STUDENTS

I was able to do funded research with two students during my postdoc at Northwestern and had incredible experiences with both students. I'm excited to continue doing research with students, and I have several ideas that could be suitable for a range of interests. Some of the projects are squarely in the field of algebraic topology and would be appropriate for a student who has taken a topology class. Other projects are geared toward those who may have less topology background, or those with coding skills. As a third option, I am also interested in projects with students that lean toward exposition on a wide variety of topics.

Algebraic topology projects

The representation spheres I worked on in my thesis are tractable geometric objects that can be drawn in low dimensions. Describing and decomposing such spheres could be a fun and enriching project.

- Dihedral groups and finite cyclic groups have irreducible representations all of dimension less than three. Students could draw some of the representation spheres associated to these representations and construct G-CW cell structures for those spheres. This kind of project would rely heavily on geometric reasoning and give students a chance to become acquainted with group actions and topological spaces.

- Some spaces are more difficult to draw than spheres but may still be simple enough to work it concretely. One possibility is projective spaces created by identifying antipodal points of representation spheres. This project can be a natural extension of the one above, or completely separate.

- Work by Hazel [Haz23] on classifying all C_2 surfaces and by Pohland [Poh22] on C_p surfaces for odd primes p could be extended to the non-abelian group of order 21. It would be a worthwhile project to construct some surfaces that are not spheres but still have an action by G_{21}. This project would introduce students to group actions and techniques for classifying and constructing manifolds.

Homotopical combinatorics projects

Many features of transfer systems can be approached combinatorially, making them well-suited to projects with undergraduates. Both [FOO+21] and [BHK+23] are examples of rich research done by undergraduates in this field, and given the newness of this topic, there are a large number of accessible projects to choose from.

- Thus far the number of transfer systems on a lattice has been enumerated for the square lattice $[p] \times [p]$ [BHK+23], the finite total order $[n]$ [BBR21], and the long lattice $[1] \times [n]$ [BMO24]. Since the definition of a transfer system can be boiled down to rules about edges of a graph, students could start by explicitly drawing transfer systems on $[2] \times [3]$ and expand to larger lattices.

- Transfer systems on a poset are related to model structures on that poset in that the acyclic fibrations of a model structure form a transfer system. One project idea would have students characterizing how to take a given transfer system and concoct every possible model structure whose acyclic fibrations match that transfer system. This project would introduce students to both combinatorics and model categories.

Computer projects

Many areas of math are enriched by suitable use of computer technology, so I have also prepared some ideas for student projects that blend the two areas.

- Grappling with 3D visualization is useful but tough, and producing computer graphics that display spheres with group action would be a great way to practice those skills. Students who are interested in producing video content could make animations of group actions on various spheres. This kind of project would allow students to learn about group actions and use computer skills to support those efforts.

- Computing classical homology and equivariant homology can in theory be done algorithmically. If a space has a known cell structure, then it is possible to compute homology in a concrete, albeit tedious way. So one interesting project for students could be to learn about this method for computing homology and write code for performing these computations. Students working on this project would learn linear algebra and algebraic topology.

Expository projects

Algebraic topology is a subfield of math that is heavy with vocabulary and constructions. Because of this I have seen REUs, like the one run by Dr. Peter May at The University of Chicago, where the goal is for students to produce thorough expositions on a single topic. For example, students interested in my work may wish to learn about model categories or Bredon homology, while students interested in more general algebraic topology may wish to write about spectra or Spanier-Whitehead duality. This kind of project can be very flexible and fit a wide range of student needs.

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